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Abstract

This paper establishes a connection between the theory of convex geometries, the principle of inclusion-exclusion and the theory of abstract tubes. In particular, it is shown that convex geometries give rise to improved inclusion-exclusion identities and via abstract tubes to improved Bonferroni inequalities. Thus, several results from the literature are rediscovered in a concise and unified way. Our general results are applied in identifying a new class of hypergraphs for which the reliability covering problem can be solved in polynomial time.

Keywords: convex geometry, inclusion-exclusion, Bonferroni inequalities, sieve formula, abstract tube, abstract simplicial complex, contractible, partition lattice, system reliability, reliability covering problem, domination theory

AMS Classification (1991): 05A15, 05A19, 05A20, 52A01, 60C05, 90B25

1 Introduction

Undoubtlessly, one of the most important tools in combinatorial probability theory and reliability theory is the principle of inclusion-exclusion and the associated Bonferroni inequalities; see Galambos and Simonelli [GS96] for a detailed account. For any finite family of sets $\{A_v\}_{v\in V}$ the principle of inclusion-exclusion states that

$$\chi\left(\bigcup_{v\in V} A_v\right) = \sum_{\substack{J\subseteq V\\ J\neq\emptyset}} (-1)^{|J|-1}\chi\left(\bigcap_{j\in J} A_j\right), \tag{1}$$

and the Bonferroni inequalities are

$$\chi\left(\bigcup_{v\in V} A_v\right) \geq \sum_{\substack{J\subseteq V, J\neq\emptyset\\ |J|\leq r}} (-1)^{|J|-1}\chi\left(\bigcap_{j\in J} A_j\right) \quad (r \text{ even}), \tag{2}$$

$$\chi\left(\bigcup_{v\in V} A_v\right) \leq \sum_{\substack{J\subseteq V, J\neq\emptyset\\ |J|\leq r}} (-1)^{|J|-1}\chi\left(\bigcap_{j\in J} A_j\right) \quad (r \text{ odd}), \tag{3}$$

where $\chi(A)$ is used to denote the indicator function of A, that is, $\chi(A)(\omega) = 1$ if $\omega \in A$, and $\chi(A)(\omega) = 0$ if $\omega \notin A$. There is no real restriction in using indicator functions rather than measures, since Eqs. (1)–(3) can be integrated with respect to any measure (e.g., a probability measure) on the algebra generated by $\{A_v\}_{v\in V}$.

Since each sum on the right-hand sides of (1)–(3) ranges over a large number of terms, it is natural to ask whether fewer terms would give the same or an even better result. Partial answers to this question have been given by several authors, e.g., McKee [McK97, McK98], Naiman and Wynn [NW92, NW97], Narushima [Nar74, Nar82] and the author [Doh99a, Doh99d]. The problem naturally arises when assessing the reliability of a network [Shi88, Shi91] or when computing the volume of a union of spherical balls or other geometric objects in Euclidean space [ER97, NW97].

This paper unifies some of the known results in the area by establishing a connection with the theory of convex geometries raised by Edelman and Jamison [EJ85]. Section 2 reviews the concept of a convex geometry and provides some examples that will be used in later sections. In Section 3 we establish an improved inclusion-exclusion identity for each finite family of sets, whose indices are the points of a convex geometry and whose sets are sufficiently well-structured. We thus rediscover several results from the literature in a concise and unified way. The section concludes with the introduction of a new type of domination, similar to domination in reliability theory. In Section 4 the identities of Section 3 are restated and generalized in terms of abstract tubes, a topological concept that was introduced by Naiman and Wynn [NW97] and that gives rise to improved Bonferroni inequalities. Although these inequalities generalize the identities of Section 3, we prefer to treat the identities separately, since they can be proved in an elementary combinatorial way. In contrast, the associated inequalities require a rather topological proof. Sections 5 and 6 provide applications to system reliability analysis and reliability covering problems.

2 Convex geometries

For any set V, we use $\mathcal{P}(V)$ to denote the set of subsets of V and $\mathcal{P}^*(V)$ to denote the set of non-empty subsets of V. A *closure operator* on V is a mapping c from $\mathcal{P}(V)$ into itself such that for any subset X of V,

$$\begin{array}{rl} \text{(i)} & X \subseteq c(X) \quad \text{(extensionality)}, \\ \text{(ii)} & X \subseteq Y \Rightarrow c(X) \subseteq c(Y) \quad \text{(monotonicity)}, \\ \text{(iii)} & c(c(X)) = c(X) \quad \text{(idempotence)}. \end{array}$$

If c is a closure operator on V, then a subset X of V is referred to as *c-closed* if $c(X) = X$ and as *c-free* if all subsets of X are c-closed. A *c-basis* of X is a minimal subset B of X such that $c(B) = X$. If there are no ambiguities, we simply write *closed* instead of c-closed, *free* instead of c-free, and *basis* instead of c-basis.

A *convex geometry* is a pair (V, c) consisting of a finite set V and a closure operator c on V such that any closed set has a unique basis. For equivalent characterizations of convex geometries, see Edelman and Jamison [EJ85]. Some examples follow.

Example 2.1 [EJ85] Let V be a finite set of points in $\mathbb{R}^d$, and for any subset X of V let $\mathrm{conv}(X)$ denote the convex hull of X, which is a polytope in $\mathbb{R}^d$. Now, by $c(X) := \mathrm{conv}(X) \cap V$ a closure operator on V is defined. By the well-known Krein-Milman theorem, any c-closed subset X of V has a unique c-basis, which consists of the vertices of the polytope $\mathrm{conv}(X)$. Thus, (V, c) is a convex geometry.

Example 2.2 [EJ85] A *simple graph* is a pair $G = (V, E)$ where V is a finite set (consisting of *vertices*) and E is a set of two-element subsets of V (consisting of *edges*). A *tree* is a simple graph which is connected and cycle-free in the usual graph-theoretic sense. For any tree $G = (V, E)$ and any subset X of V define

$$c(X) := \bigcup_{x,y \in X} \{z \in V \mid z \text{ is on the path between } x \text{ and } y\} .$$

Then, any c-closed subset X of V has a unique base, namely the set of its leaves. Thus, (V, c) is a convex geometry. Evidently, a subset X of V is free if and only if X is an edge or a singleton. The following example generalizes this one.

Example 2.3 [EJ85] A *chord* of a path P of a simple graph G is an edge of G joining two non-adjacent vertices of P. Similarly for cycles. A *chordal graph* is a simple graph G where any cycle of length greater than three has a chord. For any connected chordal graph $G = (V, E)$ and any subset X of V define

$$c(X) := \bigcup_{x,y \in X} \{z \in V \mid z \text{ is on a chordless path between } x \text{ and } y\} .$$

Then, (V, c) is a convex geometry, where the free sets are precisely the cliques of G, that is, those subsets X of V such that any two vertices of X are adjacent in G.

Example 2.4 [EJ85] An *upper semilattice* is a partially ordered set V such that any two elements $x, y \in V$ have a least common upper bound in V, which is denoted by $x \vee y$ and which is called the *supremum* of x and y in V. Dually, a *lower semilattice* is a partially ordered set V such that any two elements $x, y \in V$ have a greatest common lower bound in V, which is denoted by $x \wedge y$ and which is called the *infimum* of x and y in V. Now, let V be a finite upper (resp. lower) semilattice, and for any subset X of V let $c(X)$ be the smallest (with respect to inclusion) upper (resp. lower) subsemilattice of V including X. Then, a subset X of V is closed if and only if X is an upper (resp. lower) subsemilattice of V. Obviously, each closed subset X of V has a unique base, namely the set of all $x \in V$ such that $x = a \vee b$ (resp. $x = a \wedge b$) implies $x = a$ or $x = b$. Thus, (V, c) is a convex geometry. Note that a subset X of V is free if and only if X is a chain, that is, if any two elements of X are comparable.

The next example does not appear in the literature.

Example 2.5 Let V be a finite linearly ordered set, $\mathcal{X} \subseteq \mathcal{P}^*(V)$, $\{v_X\}_{X \in \mathcal{X}} \subseteq V$ such that $v_X > \max X$ (resp. $v_X < \min X$) for any $X \in \mathcal{X}$. For any $I \subseteq V$ define

$$\begin{aligned} c(I) &:= I \cup \{v_X \mid X \in \mathcal{X}, X \subseteq I\}, \\ c^*(I) &:= c(I) \cup c(c(I)) \cup c(c(c(I))) \cup c(c(c(c(I)))) \cup \ldots . \end{aligned}$$

Then, c^* is a closure operator on V, and evidently, any c^*-closed subset I of V has a unique c^*-basis, which is given by $I \setminus \{v_X \mid X \in \mathcal{X}, X \subseteq I\}$. Thus, (V, c^*) is a convex geometry. Obviously, a subset I of V is c^*-free if and only if $I \not\supseteq X$ for any $X \in \mathcal{X}$.

We will come back to these examples in the next section.

3 Improved inclusion-exclusion identities

We first establish some preliminary results. For any closure operator on a finite set, the following proposition characterizes the free sets by means of their bases.

Proposition 3.1 *Let V be a finite set, and let c be a closure operator on V. Then, any subset J of V is free if and only if it is a basis of its own.*

Proof. Trivially, if J is free, then J is a basis of its own. Subsequently, the opposite direction is proved by contraposition. Assume that J is not free, that is, $K \subset J$ for some non-closed set K. If J is not closed, then it is not a basis of its own, and we are done. Thus, assume that J is closed. For each $k \in c(K) \setminus K$ we find that $k \in c(K) = c(K \setminus \{k\}) \subseteq c(J \setminus \{k\}) \subseteq c(J) = J$ and hence, $J \subseteq c(J \setminus \{k\} \cup \{k\}) \subseteq c(c(J \setminus \{k\}) \cup \{k\}) = c(c(J \setminus \{k\})) = c(J \setminus \{k\}) \subseteq c(J) = J$. Therefore, $k \in J$ and $c(J \setminus \{k\}) = J$, whence J is not a base of its own. □

Although not mentioned by Edelman and Jamison [EJ85], the following proposition generalizes a result of Narushima [Nar74, Nar77] on semilattices.

Proposition 3.2 [EJ85] *For any closed set J in a convex geometry (V, c),*

$$\sum_{\substack{I \subseteq J \\ c(I)=J}} (-1)^{|I|} = \begin{cases} (-1)^{|J|} & \text{if } J \text{ is free,} \\ 0 & \text{otherwise.} \end{cases}$$

Subsequently, we give our own proof of Proposition 3.2.

Proof. Let J_0 be the unique basis of J. Then, $c(I) = J$ iff $J_0 \subseteq I \subseteq J$. Hence,

$$\sum_{\substack{I \subseteq J \\ c(I)=J}} (-1)^{|I|} = \begin{cases} (-1)^{|J|} & \text{if } J_0 = J, \\ 0 & \text{otherwise.} \end{cases}$$

From Proposition 3.1 it follows that $J_0 = J$ if and only if J is free. □

The following proposition will be used later to derive the main result of this section. It may have applications not only to inclusion-exclusion.

Proposition 3.3 *Let (V, c) be a convex geometry. Furthermore, let g be a mapping from the power set of V into an abelian group such that $g = g \circ c$. Then,*

$$\sum_{I \subseteq V} (-1)^{|I|} g(I) = \sum_{\substack{J \subseteq V \\ J \text{ free}}} (-1)^{|J|} g(J).$$

Proof. By the requirements, $g(I) = g(c(I))$ for any subset I of V. Therefore,

$$\sum_{I \subseteq V} (-1)^{|I|} g(I) = \sum_{I \subseteq V} (-1)^{|I|} g(c(I)) = \sum_{\substack{J \subseteq V \\ c(J)=J}} \sum_{\substack{I \subseteq J \\ c(I)=J}} (-1)^{|I|} g(J).$$

Now, by applying Proposition 3.2 the statement immediately follows. □

Although we do not need the following corollaries, we state them since they are interesting in their own right.

Corollary 3.4 *The number of free sets in a convex geometry (V, c) is equal to*

$$\sum_{I \subseteq V} (-1)^{|c(I) \setminus I|}.$$

Proof. For any $I \subseteq V$ define $g(I) := (-1)^{|c(I)|}$ and apply Proposition 3.3. □

Corollary 3.5 *Let (V, c) be a convex geometry. Then,*

$$\sum_{I \subseteq V} (-1)^{|I|} |c(I)| = \sum_{\substack{J \subseteq V \\ J \text{ free}}} (-1)^{|J|} |J|. \tag{4}$$

Proof. For any $I \subseteq V$ define $g(I) := |c(I)|$ and apply Proposition 3.3. □

Remark. For the convex geometry of Example 2.1, where c is derived from the convex hull operator in $\mathbb{R}^d$, Corollary 3.5 specializes to a recent result of Gordon [Gor97]. An even more recent result of Edelman and Reiner [ER] states that either side of (4) agrees in absolute value with the number of points in V which are in the interior of $\mathrm{conv}(V)$. This settles a conjecture of Ahrens, Gordon and McMahon [AGM99], who gave a proof for the two-dimensional case.

We continue with a further important proposition.

Proposition 3.6 *Let $\{A_v\}_{v \in V}$ be a finite family of sets, and let c be a closure operator on V such that for any non-empty and non-closed subset X of V,*

$$\bigcap_{x \in X} A_x \subseteq \bigcup_{v \notin X} A_v \,. \tag{5}$$

Then, for any non-empty subset I of V,

$$\bigcap_{i \in I} A_i = \bigcap_{i \in c(I)} A_i \,.$$

Proof. Fix some $I \subseteq V$, $I \neq \emptyset$. There is nothing to prove if $\bigcap_{i \in I} A_i = \emptyset$. Otherwise choose $\omega \in \bigcap_{i \in I} A_i$ and show that $\omega \in \bigcap_{i \in c(I)} A_i$. To this end, set

$$V_\omega := \{v \in V \,|\, \omega \in A_v\}$$

and by induction on n define

$$I_{n+1} := I_n \cup (V \setminus I_n) \cap V_\omega \quad (n \in \mathbb{N}); \qquad I_1 := I \,.$$

Then, $I \subseteq I_n \subseteq V_\omega$ for any $n \in \mathbb{N}$. Since V is finite and $(I_n)_{n \in \mathbb{N}}$ is increasing, there is some $n \in \mathbb{N}$ such that $I_{n+1} = I_n$ and hence, $(V \setminus I_n) \cap V_\omega = \emptyset$. From this and (5), it follows that I_n is closed and hence, $I_n \supseteq c(I)$. Therefore, $\omega \in \bigcap_{i \in I_n} A_i \subseteq \bigcap_{i \in c(I)} A_i$, and thus the proposition is proved. □

We are now ready to state the main result of this section, which is both a generalization and improvement of the classical inclusion-exclusion identity.

Theorem 3.7 *Let (V, c) be a convex geometry, and let $\{A_v\}_{v \in V}$ be a finite family of sets such that for any non-empty and non-closed subset X of V,*

$$\bigcap_{x \in X} A_x \subseteq \bigcup_{v \notin X} A_v \,.$$

Then,

$$\chi\left(\bigcup_{v \in V} A_v\right) = \sum_{\substack{J \in \mathcal{P}^*(V) \\ J \text{ free}}} (-1)^{|J|-1} \chi\left(\bigcap_{j \in J} A_j\right) .$$

Proof. By the classical inclusion-exclusion identity,

$$\chi\left(\bigcup_{v\in V} A_v\right) = \sum_{I\in\mathcal{P}^*(V)} (-1)^{|I|-1} g(I) \quad \text{where} \quad g(I) := \chi\left(\bigcap_{i\in I} A_i\right).$$

By Proposition 3.6, $g = g \circ c$. Therefore, Proposition 3.3 can be applied. □

Remarks. Note that by setting $c(X) := X$ for any subset X of V, the improved identity of Theorem 3.7 specializes to the classical inclusion-exclusion identity.

We further remark that the improved identity of Theorem 3.7 requires intersections of at most $h(c) := \max\{|J| : J \text{ } c\text{-free}\}$ sets. In [JW81] it is shown that $h(c)$ is the Helly number of the family of all c-closed subsets of V, that is, the smallest integer h such that any family of c-closed subsets of V whose intersection is empty has a subfamily of h or less sets whose intersection is also empty.

From Theorem 3.7 we now deduce some particular results, which for the first time appear in a common context. Among these results are the tree sieve of Naiman and Wynn [NW92] and the semilattice sieve of Narushima [Nar74].

Corollary 3.8 [NW92] *Let $\{A_v\}_{v\in V}$ be a finite family of sets, where the indices form the vertices of a tree $G = (V, E)$ such that $A_x \cap A_y \subseteq A_z$ for any $x, y \in V$ and any z on the path between x and y in G. Then,*

$$\chi\left(\bigcup_{v\in V} A_v\right) = \sum_{i\in V} \chi(A_i) - \sum_{\{i,j\}\in E} \chi(A_i \cap A_j).$$

Proof. Apply Theorem 3.7 in connection with Example 2.2. □

Since any tree is chordal, the following result generalizes the preceding one.

Corollary 3.9 *Let $\{A_v\}_{v\in V}$ be a finite family of sets, where the indices form the vertices of a connected chordal graph $G = (V, E)$ such that $A_x \cap A_y \subseteq A_z$ for any $x, y \in V$ and any z on any chordless path between x and y in G. Then,*

$$\chi\left(\bigcup_{v\in V} A_v\right) = \sum_{\substack{J\in\mathcal{P}^*(V) \\ J \text{ is a clique}}} (-1)^{|J|-1} \chi\left(\bigcap_{j\in J} A_j\right).$$

Proof. Apply Theorem 3.7 in connection with Example 2.3. □

Remark. Note that Corollary 3.9 specializes to the classical inclusion-exclusion identity if G is complete, that is, if any two vertices of G are adjacent.

Corollary 3.10 [Doh99a] *Let $\{A_v\}_{v\in V}$ be a finite family of sets, where V is endowed with a linear ordering relation, and let $\mathfrak{X}$ be a set of non-empty subsets of V such that for any $X \in \mathfrak{X}$,*

$$\bigcap_{x\in X} A_x \subseteq A_v \quad \textit{for some } v > \max X.$$

Then,

$$\chi\left(\bigcup_{v\in V} A_v\right) = \sum_{\substack{J\in\mathcal{P}^*(V)\\ J\not\supseteq X (\forall X\in\mathfrak{X})}} (-1)^{|J|-1}\chi\left(\bigcap_{j\in J} A_j\right).$$

Proof. Apply Theorem 3.7 in connection with Example 2.5. □

Remarks. As noted in [Doh99a], Corollary 3.10 generalizes a theorem of Whitney [Whi32]. Notice that Corollary 3.10 can be dualized by replacing $v > \max X$ with $v < \min X$, and that it reduces to the classical inclusion-exclusion identity if $\mathfrak{X} = \emptyset$.

From Corollary 3.10 we now deduce the following inclusion-exclusion variant, which in a slightly less general form was first established by Narushima [Nar82].

Corollary 3.11 [Nar82] *Let $\{A_v\}_{v\in V}$ be a finite family of sets, where V is endowed with a partial ordering relation such that for any $x, y \in V$, $A_x \cap A_y \subseteq A_z$ for some upper bound z of x and y. Then,*

$$\chi\left(\bigcup_{v\in V} A_v\right) = \sum_{\substack{J\in\mathcal{P}^*(V)\\ J \text{ is a chain}}} (-1)^{|J|-1}\chi\left(\bigcap_{j\in J} A_j\right). \tag{6}$$

Proof. Corollary 3.11 follows from Corollary 3.10 by defining $\mathfrak{X}$ as the set of all unordered pairs of incomparable elements of V and then considering an arbitrary linear extension of the partial ordering relation on V. □

Remarks. Since the chains of a partially ordered set V are the cliques of its comparability graph $G_V := \big(V, \{\{v,w\} \mid v < w \text{ or } w < v\}\big)$, the sum in (6) may be viewed as being extended over all non-empty cliques of the comparability graph of V. This raises the question whether Corollary 3.11 can also be deduced from Corollary 3.9. This is seemingly not the case since the comparability graph need not be chordal.

The reader should note that the requirements of Corollary 3.11 are weaker than the original requirements of Narushima [Nar82]. Namely, Narushima [Nar82] requires that for any $x, y \in V$, $A_x \cap A_y \subseteq A_z$ for some *minimal* upper bound z of x and y. In Corollary 3.11, however, the minimality of z is not required. Thus, Corollary 3.11 is more general than Narushima's original result [Nar82].

Corollary 3.11 specializes to the classical inclusion-exclusion identity if the partial ordering relation on V is linear, or in other words, if V is a chain. In the extreme case where V has a maximum $\hat{1}$ and any distinct $x, y < \hat{1}$ are incomparable and satisfy $A_x \cap A_y \subseteq A_{\hat{1}}$, Corollary 3.11 requires evaluation of only $2|V| - 1$ terms, whereas the traditional inclusion-exclusion principle would require evaluation of $2^{|V|} - 1$ terms.

The requirements of Corollary 3.11 are already satisfied if V is a finite upper semilattice and $A_x \cap A_y \subseteq A_{x \vee y}$ for any $x, y \in V$. Thus, Corollary 3.11 specializes to the semilattice sieve of Narushima [Nar74], which can also be obtained by applying Theorem 3.7 in connection with the convex geometry of Example 2.4.

We close this section with a generalization of Proposition 3.2, Proposition 3.3 and Theorem 3.7. Let V be a finite set, and let c be a closure operator on V. A *c-formation* of a subset X of V is any non-empty set $\mathcal{B}$ of c-bases of X such that $\bigcup \mathcal{B} = X$. A c-formation $\mathcal{B}$ of X is *odd* resp. *even* if $|\mathcal{B}|$ is odd resp. even. The *c-domination* of X, $\mathrm{dom}_c(X)$, is the number of odd c-formations of X minus the number of even c-formations of X. Evidently, if X is not c-closed, then $\mathrm{dom}_c(X) = 0$, and if X is c-free, then $\mathrm{dom}_c(X) = 1$. If (V, c) is a convex geometry, then $\mathrm{dom}_c(X) = 1$ resp. 0 depending on whether X is c-free or not.

Proposition 3.12 *Let V be a finite set, and let c be a closure operator on V. Then, for any c-closed subset J of V,*

$$\sum_{\substack{I \subseteq J \\ c(I) = J}} (-1)^{|I|} = (-1)^{|J|} \mathrm{dom}_c(J) .$$

Proof. Let $J_0, \ldots, J_n$ be the distinct bases of J. Evidently, $c(I) = J$ if and only if $J_k \subseteq I \subseteq J$ for some $k \in \{0, \ldots, n\}$. Thus, by traditional inclusion-exclusion,

$$\sum_{\substack{I \subseteq J \\ c(I) = J}} (-1)^{|I|} = \sum_{\substack{\mathcal{B} \subseteq \{J_0, \ldots, J_n\} \\ \mathcal{B} \neq \emptyset}} (-1)^{|\mathcal{B}|-1} \sum_{I : \bigcup \mathcal{B} \subseteq I \subseteq J} (-1)^{|I|} = \sum_{\substack{\mathcal{B} \subseteq \{J_0, \ldots, J_n\} \\ \mathcal{B} \neq \emptyset, \bigcup \mathcal{B} = J}} (-1)^{|\mathcal{B}|-1} (-1)^{|J|} .$$

The result now follows from the definition of the c-domination $\mathrm{dom}_c(J)$. □

Proposition 3.13 *Let V be a finite set, c a closure operator on V and g a mapping from $\mathcal{P}(V)$ into an abelian group such that $g = g \circ c$. Then,*

$$\sum_{I \subseteq V} (-1)^{|I|} g(I) = \sum_{\substack{J \subseteq V \\ J \text{ closed}}} (-1)^{|J|} \mathrm{dom}_c(J)\, g(J) .$$

Proof. Proposition 3.13 follows from Proposition 3.12 in the same way as Proposition 3.3 follows from Proposition 3.2. □

Theorem 3.14 *Let $\{A_v\}_{v \in V}$ be a finite family of sets, and let c be a closure operator on V such that for any non-empty and non-closed subset X of V,*

$$\bigcap_{x \in X} A_x \subseteq \bigcup_{v \notin X} A_v .$$

Then,

$$\chi\left(\bigcup_{v\in V} A_v\right) = \sum_{\substack{J\in\mathcal{P}^*(V)\\ J\ closed}} (-1)^{|J|-1}\,\mathrm{dom}_c(J)\,\chi\left(\bigcap_{j\in J} A_j\right).$$

Proof. Theorem 3.14 follows from Proposition 3.13 and Proposition 3.6 in the same way as Theorem 3.7 follows from Proposition 3.3 and Proposition 3.6. □

4 Improved Bonferroni inequalities

The results of this section require some basic knowledge of combinatorial topology. In order to keep the exposition self-contained, we briefly review the necessary facts. For further reading, the reader is referred to the textbook of Harzheim [Har78].

An *abstract simplicial complex* $\mathcal{S}$ is a set of non-empty subsets of some finite set V such that $I \in \mathcal{S}$ and $\emptyset \neq J \subset I$ imply $J \in \mathcal{S}$. The elements of $\mathcal{S}$ are the *faces* or *simplices* of $\mathcal{S}$, whereas the elements of $\mathrm{Vert}(\mathcal{S}) := \bigcup_{I\in\mathcal{S}} I$ are the *vertices* of $\mathcal{S}$. The *dimension* of a face I, $\dim I$, is one less than its cardinality. The *dimension* of $\mathcal{S}$, $\dim\mathcal{S}$, is the maximum dimension of a face in $\mathcal{S}$. A *geometric realization* of $\mathcal{S}$ is any topological space homeomorphic to

$$|\mathcal{S}| := \bigcup_{I\in\mathcal{S}} \mathrm{conv}(\{\mathbf{e}_i \mid i \in I\}), \tag{7}$$

where $\{\mathbf{e}_v\}_{v\in V}$ is the standard basis of $\mathbb{R}^V$ and $V = \mathrm{Vert}(\mathcal{S})$. Recall that two topological spaces X and Y are *homeomorphic* if there exists a bijective mapping $\phi : X \to Y$ such that both ϕ and its inverse ϕ^{-1} are continuous. Evidently, a geometric realization is unique up to homeomorphism.

A topological space X is *contractible* if there is a continuous map $F : X \times [0,1] \to X$ such that $F(x,0) = x$ for any $x \in X$ and $F(\cdot\,,1) \equiv c$ for some constant $c \in X$. Since contractibility is known to be a homeomorphism invariant, we may call an abstract simplicial complex *contractible* if it has a contractible geometric realization. For example, for any non-empty finite set V the abstract simplicial complex $\mathcal{P}^*(V)$ consisting of all non-empty subsets of V is contractible.

Following Naiman and Wynn [NW97], an *abstract tube* is a pair $(\mathcal{A}, \mathcal{S})$ consisting of a finite collection of sets $\mathcal{A} = \{A_v\}_{v\in V}$ and an abstract simplicial complex $\mathcal{S} \subseteq \mathcal{P}^*(V)$ such that for any $\omega \in \bigcup_{v\in V} A_v$ the abstract simplicial complex

$$\mathcal{S}(\omega) := \left\{ I \in \mathcal{S} \,\middle|\, \omega \in \bigcap_{i\in I} A_i \right\}$$

is contractible. Given two abstract tubes $(\mathcal{A}_1, \mathcal{S}_1)$ and $(\mathcal{A}_2, \mathcal{S}_2)$, we say that $(\mathcal{A}_1, \mathcal{S}_1)$ is a *subtube* of $(\mathcal{A}_2, \mathcal{S}_2)$ if $\mathcal{A}_1 = \mathcal{A}_2$ and $\mathcal{S}_1 \subseteq \mathcal{S}_2$.

In the following, we restate the main results of abstract tube theory due to Naiman and Wynn [NW97] without proof.

Proposition 4.1 [NW97] *Let* $(\{A_v\}_{v\in V}, \mathcal{S})$ *be an abstract tube. Then, for* $r \in \mathbb{N}$,

$$\chi\left(\bigcup_{v\in V} A_v\right) \geq \sum_{\substack{I\in\mathcal{S}\\|I|\leq r}} (-1)^{|I|-1}\chi\left(\bigcap_{i\in I} A_i\right) \qquad (r\ \textit{even}),$$

$$\chi\left(\bigcup_{v\in V} A_v\right) \leq \sum_{\substack{I\in\mathcal{S}\\|I|\leq r}} (-1)^{|I|-1}\chi\left(\bigcap_{i\in I} A_i\right) \qquad (r\ \textit{odd}).$$

Proposition 4.2 [NW97] *Let* $(\{A_v\}_{v\in V}, \mathcal{S})$ *and* $(\{A_v\}_{v\in V}, \mathcal{S}')$ *be abstract tubes, where* $(\{A_v\}_{v\in V}, \mathcal{S}')$ *is a subtube of* $(\{A_v\}_{v\in V}, \mathcal{S})$. *Then, for any* $r \in \mathbb{N}$,

$$\sum_{\substack{I\in\mathcal{S}'\\|I|\leq r}} (-1)^{|I|-1}\chi\left(\bigcap_{i\in I} A_i\right) \geq \sum_{\substack{I\in\mathcal{S}\\|I|\leq r}} (-1)^{|I|-1}\chi\left(\bigcap_{i\in I} A_i\right) \qquad (r\ \textit{even}),$$

$$\sum_{\substack{I\in\mathcal{S}'\\|I|\leq r}} (-1)^{|I|-1}\chi\left(\bigcap_{i\in I} A_i\right) \leq \sum_{\substack{I\in\mathcal{S}\\|I|\leq r}} (-1)^{|I|-1}\chi\left(\bigcap_{i\in I} A_i\right) \qquad (r\ \textit{odd}).$$

Remarks. Since $(\{A_v\}_{v\in V}, \mathcal{P}^*(V))$ is an abstract tube for any finite collection of sets $\{A_v\}_{v\in V}$, the classical Bonferroni inequalities are a particular case of Proposition 4.1. Moreover, since any abstract tube $(\{A_v\}_{v\in V}, \mathcal{S})$ is a subtube of $(\{A_v\}_{v\in V}, \mathcal{P}^*(V))$, Proposition 4.2 especially states that the bounds provided by Proposition 4.1 are at least as sharp as their classical counterparts, although less computational effort is needed to compute them. We further remark that the inequalities in Proposition 4.1 become an identity if $r \geq \dim \mathcal{S} + 1$. In particular, any abstract tube $(\{A_v\}_{v\in V}, \mathcal{S})$ gives rise to an improved inclusion-exclusion identity for the indicator function of $\bigcup_{v\in V} A_v$ which does not require intersections of more than $\dim \mathcal{S}+1$ sets, that is, the most complicated intersection is $(\dim \mathcal{S}+1)$-fold. Thus, in the terminology of Naiman and Wynn [NW97], any abstract tube $(\mathcal{A}, \mathcal{S})$ gives rise to an inclusion-exclusion identity of depth $\dim \mathcal{S} + 1$.

Due to Naiman and Wynn [NW97], the definition of an abstract tube can be weakened by requiring contractibility of $\mathcal{S}(\omega)$ for almost every ω with respect to some dominating measure μ on the ambient space. In this case, the improved Bonferroni inequalities of Proposition 4.1 and Proposition 4.2 (and the associated inclusion-exclusion identities) hold almost everywhere with respect to μ, and the pair $(\mathcal{A}, \mathcal{S})$ is referred to as a *weak abstract tube*. If μ is a probability measure, then the mapping $\omega \mapsto \mathcal{S}(\omega)$ may be considered as a random abstract simplicial complex $\mathcal{S}^{\text{ran}}$ which is required to be almost surely contractible.

Subsequently, our improved inclusion-exclusion identities of Section 3 are restated and generalized in terms of abstract tubes. Recall from the above that any abstract tube gives rise to a series of improved Bonferroni inequalities. We do not mention these inequalities explicitly, since they can easily be read from Proposition 4.1.

Our first result is an abstract tube generalization of Theorem 3.7. For any convex geometry (V, c), we use $\mathrm{Free}(V, c)$ to denote the set of all non-empty c-free subsets of V. Obviously, $\mathrm{Free}(V, c)$ is an abstract simplicial complex.

Theorem 4.3 *Let (V, c) be a convex geometry, and let $\{A_v\}_{v\in V}$ be a finite family of sets such that for any non-empty and non-closed subset X of V,*

$$\bigcap_{x\in X} A_x \subseteq \bigcup_{v\notin X} A_v .$$

Then, $(\{A_v\}_{v\in V}, \mathrm{Free}(V, c))$ is an abstract tube.

The proof of Theorem 4.3 is based on the following observation of Björner and Ziegler [BZ92, Exercise 8.23c]. For a rigorous proof of this observation the reader is referred to the very recent paper of Edelman and Reiner [ER].

Proposition 4.4 [BZ92] $\mathrm{Free}(V, c)$ *is contractible for any convex geometry* (V, c).

Proof of Theorem 4.3. Let $\omega \in \bigcup_{v\in V} A_v$, $V_\omega := \{v \in V \mid \omega \in A_v\}$ and $c_\omega(I) := c(I)$ for any $I \subseteq V_\omega$. By the definition of V_ω and the requirements of the theorem, V_ω is c-closed. Thus, (V_ω, c_ω) is a convex geometry. Since moreover $\mathrm{Free}(V, c)(\omega) = \mathrm{Free}(V_\omega, c_\omega)$, the contractibility of $\mathrm{Free}(V, c)(\omega)$ follows from Proposition 4.4. □

Remarks. In view of the remarks following Proposition 4.2, it is equally easy to prove that $(\{A_v\}_{v\in V}, \mathrm{Free}(V, c))$ is a weak abstract tube with respect to any probability measure μ on the algebra generated by $\{A_v\}_{v\in V}$ such that

$$\mu\left(\bigcap_{x\in X} A_x\right) > 0 \quad \text{and} \quad \mu\left(\bigcup_{v\notin X} A_v \,\middle|\, \bigcap_{x\in X} A_x\right) = 1$$

for any non-empty and non-closed subset X of V.

We further remark that the abstract tube $(\{A_v\}_{v\in V}, \mathrm{Free}(V, c'))$ is a subtube of $(\{A_v\}_{v\in V}, \mathrm{Free}(V, c))$ if both c and c' satisfy the requirements of Theorem 4.3 and $c' \leq c$, where the partial ordering relation $\leq$ is defined by

$$c' \leq c \quad :\Leftrightarrow \quad c(I) \subseteq c'(I) \text{ for any subset } I \text{ of } V$$

or equivalently,

$$c' \leq c \quad :\Leftrightarrow \quad \text{all } c'\text{-closed subsets of } V \text{ are } c\text{-closed.}$$

By this and Proposition 4.2, it follows that the improved Bonferroni inequalities associated with c' are at least as sharp as those associated with c if $c' \leq c$.

As a consequence of Theorem 4.3 and as an extension of Corollary 3.9 we now deduce the following result on the clique complex of a connected chordal graph G.

Corollary 4.5 *Let $\{A_v\}_{v\in V}$ be a finite family of sets, where the indices form the vertices of a connected chordal graph $G = (V, E)$ such that $A_x \cap A_y \subseteq A_z$ for any $x, y \in V$ and any z on any chordless path between x and y. Then, $\{A_v\}_{v\in V}$ together with the collection of non-empty cliques of G forms an abstract tube.*

Proof. Apply Theorem 4.3 in connection with Example 2.3. □

Corollary 4.6 [Doh99c] *Let $\{A_v\}_{v\in V}$ be a finite family of sets, where V is endowed with a linear ordering relation, and let $\mathcal{X}$ be a set of non-empty subsets of V such that for any $X \in \mathcal{X}$,*

$$\bigcap_{x\in X} A_x \subseteq A_v \quad \textit{for some } v > \max X.$$

Then, $\left(\{A_v\}_{v\in V}, \{I \in \mathcal{P}^*(V) \mid I \not\supseteq X\, \forall X \in \mathcal{X}\}\right)$ is an abstract tube.

Proof. Apply Theorem 4.3 in connection with Example 2.5. □

For any partially ordered set V we use $\mathcal{C}(V)$ to denote the *order complex* of V, that is, the abstract simplicial complex consisting of all non-empty chains of V.

Corollary 4.7 [Doh99c] *Let $\{A_v\}_{v\in V}$ be a finite family of sets, where V is endowed with a partial ordering relation such that for any $x, y \in V$, $A_x \cap A_y \subseteq A_z$ for some upper bound z of x and y. Then, $(\{A_v\}_{v\in V}, \mathcal{C}(V))$ is an abstract tube.*

Proof. Corollary 4.7 follows from Corollary 4.6 in the same way as Corollary 3.11 follows from Corollary 3.10. □

Remark. Note that the requirements of Corollary 4.7 are already satisfied if V is an upper semilattice such that $A_x \cap A_y \subseteq A_{x\vee y}$ for any $x, y \in V$. In this way, a specialization of Corollary 4.7 is obtained, which can also be deduced from Theorem 4.3 in connection with the convex geometry of Example 2.4.

We close this section with a generalization and restatement of Narushima's inclusion-exclusion identity for partition lattices [Nar74], which turned out as a very useful tool in the enumeration of non-isomorphic reduced finite automata [Nar77].

We review some familiar definitions. Let S be a set. A *partition* of S is a set of non-empty and pairwise disjoint subsets of S whose union is S. Each element of the partition is referred to as a *block* of the partition. The set $\Pi(S)$ of all partitions of S is given the structure of a lattice by imposing for any $\pi, \tau \in \Pi(S)$,

$$\pi \leq \tau \;:\Leftrightarrow\; \text{each block of } \pi \text{ is included by a block of } \tau.$$

For $s, s' \in S$ we write $s\pi s'$ if s and s' belong to the same block of π. Thus, $\pi \leq \tau$ if and only if for any $s, s' \in S$, $s\pi s'$ entrains $s\tau s'$. For any sets S and T, the cartesian product $\Pi(S) \times \Pi(T)$ is given the structure of a lattice by imposing

$$(\pi_1, \tau_1) \leq (\pi_2, \tau_2) \;:\Leftrightarrow\; \pi_1 \leq \pi_2 \text{ and } \tau_1 \leq \tau_2$$

for any $(\pi_1, \tau_1), (\pi_2, \tau_2) \in \Pi(S) \times \Pi(T)$.

The following definitions are non-standard: A *partitioned set* is a pair (S, π), consisting of a set S and a partition π of S. Given two partitioned sets (S, π) and (T, τ), a mapping $f : S \to T$ is called a *homomorphism from* (S, π) *to* (T, τ) if for any $s, s' \in S$, $s\pi s'$ entrains $f(s)\tau f(s')$. A homomorphism from (S, π) to itself is also referred to as an *endomorphism of* (S, π). For abbreviation, we write $f : (S, \pi) \to (T, \tau)$ if f is a homomorphism from (S, π) to (T, τ), and define

$$\begin{aligned} \mathrm{Hom}((S,\pi),(T,\tau)) &:= \{f \mid f : (S,\pi) \to (T,\tau)\}, \\ \mathrm{End}(S,\pi) &:= \mathrm{Hom}((S,\pi),(S,\pi)). \end{aligned}$$

Note that the improved inclusion-exclusion identities corresponding to the abstract tubes of the following corollaries are due to Narushima [Nar74], whereas the improved inequalities are new. Recall that $\mathcal{C}(L)$ denotes the order complex of L.

Corollary 4.8 *Let S and T be finite sets, and let L be a subsemilattice of $\Pi(S) \times \Pi(T)$. Then, $\left(\{\mathrm{Hom}((S,\pi),(T,\tau))\}_{(\pi,\tau)\in L}, \mathcal{C}(L)\right)$ is an abstract tube.*

Proof. It is easy to verify (cf. [Nar74]) that for any $(\pi, \tau), (\pi', \tau') \in L$,

$$\begin{aligned} \mathrm{Hom}((S,\pi),(T,\tau)) \cap \mathrm{Hom}((S,\pi'),(T,\tau')) &\subseteq \mathrm{Hom}((S,\pi\wedge\pi'),(T,\tau\wedge\tau')), \\ \mathrm{Hom}((S,\pi),(T,\tau)) \cap \mathrm{Hom}((S,\pi'),(T,\tau')) &\subseteq \mathrm{Hom}((S,\pi\vee\pi'),(T,\tau\vee\tau')), \end{aligned}$$

where $\wedge$ and $\vee$ stand for the infimum (greatest lower bound) and supremum (least upper bound) in $\Pi(S)$ and $\Pi(T)$. By Corollary 4.7, the result follows. □

Corollary 4.9 *Let S be a finite set, and let L be a subsemilattice of $\Pi(S)$. Then, $(\{\mathrm{End}(S,\pi)\}_{\pi\in L}, \mathcal{C}(L))$ is an abstract tube.*

Proof. Since $\pi \mapsto (\pi, \pi)$ is a lattice isomorphism from $\Pi(S)$ to $\Pi(S) \times \Pi(S)$, Corollary 4.9 follows from Corollary 4.8 by considering the diagonal case $S = T$. □

5 Application to system reliability analysis

In this chapter, we briefly describe an application of our inclusion-exclusion results to general system reliability analysis. For some concrete applications in the context of network reliability analysis, the reader is referred to [Doh98, Doh99b, Doh99c].

A *coherent binary system* is a couple $\Sigma = (E, \phi)$ consisting of a finite set E and a function ϕ from the power set of E into $\{0; 1\}$ such that $\phi(\emptyset) = 0$, $\phi(E) = 1$ and $\phi(X) \leq \phi(Y)$ for any $X, Y \subseteq E$ with $X \subseteq Y$. E and ϕ are respectively called the *component set* and the *structure function* of Σ.

At any instant of time, each component e of Σ assumes randomly and independently one of two states, *operating* or *failing*, with probabilities p_e and $q_e = 1 - p_e$, respectively. Σ is said to be *operating* resp. *failing* if ϕ applied to the set of operating components, which is also referred to as the *state* of Σ, gives 1 resp. 0. The *reliability* of Σ is the probability that Σ is operating. Since this quantity is determined by Σ and the vector of operation probabilities $\mathbf{p} = (p_e)_{e \in E}$, it is abbreviated to $\mathrm{Rel}_\Sigma(\mathbf{p})$.

A key role in calculating $\mathrm{Rel}_\Sigma(\mathbf{p})$ is played by the minpaths and mincuts of Σ: A *minpath* of Σ is a minimal set $P \subseteq E$ such that $\phi(P) = 1$; that is, $\phi(P) = 1$ and $\phi(Q) = 0$ for any proper subset Q of P. A *mincut* of Σ is a minimal set $C \subseteq E$ such that $\phi(E \setminus C) = 0$; that is, $\phi(E \setminus C) = 0$ and $\phi(E \setminus D) = 1$ for any proper subset D of C. Thus, with $\mathcal{F}$ denoting the set of minpaths resp. mincuts, we have

$$(8)\quad \mathrm{Rel}_\Sigma(\mathbf{p}) = \Pr\left(\bigcup_{F \in \mathcal{F}} \{F \text{ operates}\}\right) \quad \text{resp.} \quad 1 - \mathrm{Rel}_\Sigma(\mathbf{p}) = \Pr\left(\bigcup_{F \in \mathcal{F}} \{F \text{ fails}\}\right),$$

where Pr denotes the induced probability measure on the set of system states.

If F is a set of components, then F is said to *operate* resp. *fail* if all components in F operate resp. fail. In connection with Proposition 4.1 the first part of the following theorem yields improved inclusion-exclusion identities and Bonferroni inequalities for the right-hand sides of (8) and thus for $\mathrm{Rel}_\Sigma(\mathbf{p})$. We do not mention the identities explicitly, since they are an immediate consequence of the corresponding inequalities.

Theorem 5.1 *Let $\Sigma = (E, \phi)$ be a coherent binary system, whose set of minpaths resp. mincuts $\mathcal{F}$ is endowed with a closure operator c such that $(\mathcal{F}, c)$ is a convex geometry and such that $Y \subseteq \bigcup \mathcal{X}$ for any non-empty $\mathcal{X} \subseteq \mathcal{F}$ and any $Y \in c(\mathcal{X})$. Then,*

$$\left(\{\{F \textit{ operates}\}\}_{F \in \mathcal{F}}, \mathrm{Free}(\mathcal{F}, c)\right) \quad \textit{resp.} \quad \left(\{\{F \textit{ fails}\}\}_{F \in \mathcal{F}}, \mathrm{Free}(\mathcal{F}, c)\right)$$

is an abstract tube. In particular, in case that $\mathcal{F}$ denotes the set of minpaths,

$$\mathrm{Rel}_\Sigma(\mathbf{p}) \geq \sum_{\substack{\mathcal{I} \in \mathrm{Free}(\mathcal{F}, c) \\ |\mathcal{I}| \leq r}} (-1)^{|\mathcal{I}|-1} \prod_{e \in \bigcup \mathcal{I}} p_e \quad (r \textit{ even}).$$

$$\mathrm{Rel}_\Sigma(\mathbf{p}) \leq \sum_{\substack{\mathcal{I} \in \mathrm{Free}(\mathcal{F}, c) \\ |\mathcal{I}| \leq r}} (-1)^{|\mathcal{I}|-1} \prod_{e \in \bigcup \mathcal{I}} p_e \quad (r \textit{ odd}),$$

and in case that $\mathcal{F}$ denotes the set of mincuts,

$$1 - \mathrm{Rel}_\Sigma(\mathbf{p}) \geq \sum_{\substack{\mathcal{I} \in \mathrm{Free}(\mathcal{F}, c) \\ |\mathcal{I}| \leq r}} (-1)^{|\mathcal{I}|-1} \prod_{e \in \bigcup \mathcal{I}} q_e \quad (r \textit{ even}),$$

$$1 - \mathrm{Rel}_\Sigma(\mathbf{p}) \leq \sum_{\substack{\mathcal{I} \in \mathrm{Free}(\mathcal{F}, c) \\ |\mathcal{I}| \leq r}} (-1)^{|\mathcal{I}|-1} \prod_{e \in \bigcup \mathcal{I}} q_e \quad (r \textit{ odd}).$$

Proof. The first part follows from Theorem 4.3 with $V := \mathcal{F}$ and $A_F := \{F \text{ operates}\}$ resp. $A_F := \{F \text{ fails}\}$ for any $F \in \mathcal{F}$. The second part is an immediate consequence of the first part and Proposition 4.1. □

Remark. Note that the inequalities of Theorem 5.1 specialize to the usual Bonferroni inequalities for system reliability if $c(\mathcal{X}) = \mathcal{X}$ for any $\mathcal{X} \subseteq \mathcal{F}$. For the convex geometry of Example 2.4, where $\mathcal{F}$ is a lower (resp. upper) semilattice, the requirements of Theorem 5.1 are equivalent to $X \wedge Y \subseteq X \cup Y$ (resp. $X \vee Y \subseteq X \cup Y$) for any $X, Y \in \mathcal{F}$. Note in this case the free sets are the chains of $\mathcal{F}$. We thus rediscover Shier's chain formula for the reliability of a coherent binary system [Shi88, Shi91] as well as the corresponding inequalities which are discussed in [Doh99b, Doh99c].

The rest of this section is devoted to general domination theory (cf. [Man90]).

Let $\Sigma = (E, \phi)$ be a coherent binary system, whose set of minpaths resp. mincuts is denoted by $\mathcal{F}$. An *$\mathcal{F}$-formation* of a subset X of E is any subset $\mathcal{I}$ of $\mathcal{F}$ such that $\bigcup \mathcal{I} = X$. An $\mathcal{F}$-formation $\mathcal{I}$ of X is *odd* resp. *even* if $|\mathcal{I}|$ is odd resp. even. The *$\mathcal{F}$-domination* of X, $\mathrm{dom}_{\mathcal{F}}(X)$, is the number of odd $\mathcal{F}$-formations of X minus the number of even $\mathcal{F}$-formations of X. By the traditional inclusion-exclusion principle,

$$\mathrm{Rel}_\Sigma(\mathbf{p}) = \sum_{X \in \mathcal{P}^*(E)} \mathrm{dom}_{\mathcal{F}}(X) \prod_{e \in X} p_e \quad \text{resp.} \quad 1 - \mathrm{Rel}_\Sigma(\mathbf{p}) = \sum_{X \in \mathcal{P}^*(E)} \mathrm{dom}_{\mathcal{F}}(X) \prod_{e \in X} q_e\,.$$

The $\mathcal{F}$-domination and the c-domination of Section 3 are related as follows:

Theorem 5.2 *Let $\Sigma = (E, \phi)$ be a coherent binary system, whose set of minpaths resp. mincuts is denoted by $\mathcal{F}$, and let $c_{\mathcal{F}}$ denote the closure operator $\mathcal{I} \mapsto \{F \in \mathcal{F} \mid F \subseteq \bigcup \mathcal{I}\}$ on the power set of $\mathcal{F}$. Then, for any $X \in \{\bigcup \mathcal{I} \mid \mathcal{I} \in \mathcal{P}^*(\mathcal{F})\}$,*

$$\mathrm{dom}_{\mathcal{F}}(X) = (-1)^{|\mathcal{F}|X|-1} \mathrm{dom}_{c_{\mathcal{F}}}(\mathcal{F}|X)\,,$$

where

$$\mathcal{F}|X := \{F \in \mathcal{F} \mid F \subseteq X\}\,.$$

Proof. Evidently, $\mathcal{F}|X$ is $c_{\mathcal{F}}$-closed. Hence, by Proposition 3.12, we obtain

$$\sum_{\substack{\mathcal{I} \subseteq \mathcal{F}|X \\ c_{\mathcal{F}}(\mathcal{I}) = \mathcal{F}|X}} (-1)^{|\mathcal{I}|-1} = (-1)^{|\mathcal{F}|X|-1} \mathrm{dom}_{c_{\mathcal{F}}}(\mathcal{F}|X)\,.$$

Now, $c_{\mathcal{F}}(\mathcal{I}) = \mathcal{F}|X$ if and only if $\mathcal{I}$ is an $\mathcal{F}$-formation of X. Thus, the left-hand side of the preceding equation coincides with the $\mathcal{F}$-domination of X. □

6 Application to reliability covering problems

Reliability covering problems were introduced by Ball, Provan and Shier [BPS91] (see also [Shi91]) in order to generalize several types of reliability problems. They serve e.g. as a model for mass transit systems with reliable stops and unreliable routes. The overall reliability of such a system is the probability that each stop is served by an operating route. Further examples include evaluating the reliability of flight schedules for aircraft [BPS91] and determining the reliability of maintaining continuous surveillance of a critical point of a country's border [Shi91].

Reliability covering problems can be adequately formulated using the terminology of hypergraphs: A *hypergraph* is a couple $H = (V, \mathcal{E})$ where V is a finite set and $\mathcal{E}$ is a set of subsets of V. The elements of V and $\mathcal{E}$ are the *vertices* and *edges* of H, respectively. Thus, in case of a mass transit system, the vertices correspond to the stops and the edges to the routes of the system. Throughout, we assume that the vertices of the hypergraph are perfectly reliable, whereas the edges are subject to random and independent failure. The edge operation probabilities are given by a vector $\mathbf{p} = (p_E)_{E\in\mathcal{E}} \in [0,1]^{\mathcal{E}}$. A *covering* of V is a subset $\mathcal{X}$ of $\mathcal{E}$ such that $\bigcup \mathcal{X} = V$. Thus, in case of a mass transit system, the coverings correspond to sets of routes such that each stop is served by a route. The general objective is to compute $\operatorname{Cov}(H;\mathbf{p})$, the probability that the vertex-set of H is covered by the operating edges of H. With $\mathcal{E}(v) := \{E \in \mathcal{E} \,|\, v \in E\}$ $(v \in V)$, this coverage probability can be expressed as

$$(9) \qquad \operatorname{Cov}(H;\mathbf{p}) \;=\; 1 - \Pr\left(\bigcup_{v\in V}\bigcap_{E\in\mathcal{E}(v)} \{E \text{ fails}\}\right).$$

The following theorem provides improved inclusion-exclusion identities and improved Bonferroni inequalities for the right-hand side of (9) and thus for $\operatorname{Cov}(H;\mathbf{p})$. Again, we do not mention the improved inclusion-exclusion identities explicitly, since they are an immediate consequence of the corresponding improved inequalities.

Theorem 6.1 *Let $H = (V, \mathcal{E})$ be a hypergraph whose edges fail randomly and independently and whose vertex-set V is endowed with a closure operator c such that (V, c) is a convex geometry and such that the complement of each edge is c-closed. Then,*

$$\left(\left\{\bigcap\nolimits_{E\in\mathcal{E}(v)} \{E \textit{ fails}\}\right\}_{v\in V}, \operatorname{Free}(V,c)\right)$$

is an abstract tube. In particular, for any $\mathbf{p} = (p_E)_{E\in\mathcal{E}} \in [0,1]^{\mathcal{E}}$ and any $r \in \mathbb{N}$,

$$\operatorname{Cov}(H;\mathbf{p}) \;\le\; \sum_{\substack{I\subseteq V, |I|\le r\\ I \text{ is } c\text{-free}}} (-1)^{|I|} \prod_{\substack{E\in\mathcal{E}\\ E\cap I\neq\emptyset}} q_E \qquad (r \textit{ even}),$$

$$\operatorname{Cov}(H;\mathbf{p}) \;\ge\; \sum_{\substack{I\subseteq V, |I|\le r\\ I \text{ is } c\text{-free}}} (-1)^{|I|} \prod_{\substack{E\in\mathcal{E}\\ E\cap I\neq\emptyset}} q_E \qquad (r \textit{ odd}),$$

where $q_E = 1 - p_E$ *for any* $E \in \mathcal{E}$.

Proof. For any $v \in V$ define $A_v := \bigcap_{E \in \mathcal{E}(v)} \{E \text{ fails}\}$. Since the complement of each edge is c-closed, it follows that $\bigcap_{x \in X} A_x \subseteq A_v$ for any non-empty subset X of V and any $v \in c(X)$. Now the first part of the theorem follows from Theorem 4.3. The second part is an immediate consequence of the first part and Proposition 4.1. □

Due to Ball, Provan and Shier [BPS91], the reliability covering problem, that is, the problem of computing $\mathrm{Cov}(H; \mathbf{p})$ for given H and $\mathbf{p}$, is $\#P$-hard, even when restricted to the class of hypergraphs whose vertices are the vertices of an undirected tree and whose edges are paths of cardinality 3 in the tree (viewing paths as sets of vertices). Considering complements of paths instead of paths, or more generally, complements of subtrees instead of paths, we obtain the following positive result:

Theorem 6.2 *For hypergraphs whose vertices are the vertices of an undirected tree and whose edges are complements of subtrees of the tree, the coverage probability can be computed in polynomial time from a knowledge of the hypergraph and the tree.*

Proof. Let $G = (V, T)$ be a tree and $H = (V, \mathcal{E})$ be a hypergraph where each edge of H is the complement of a subtree of G. By combining Theorem 6.1 with Example 2.2 (or by applying Corollary 3.8) we are led to the improved inclusion-exclusion identity

$$\mathrm{Cov}(H; \mathbf{p}) = 1 - \sum_{v \in V} \prod_{E \in \mathcal{E}(v)} q_E + \sum_{\{v,w\} \in T} \prod_{E \in \mathcal{E}(v) \cup \mathcal{E}(w)} q_E ,$$

which requires $O(|V| \cdot |\mathcal{E}|)$ time. □

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