HUMBOLDT-UNIVERSITÄT ZU BERLIN

Improved Bonferroni Bounds for the Reliability of a Communication Network

K. Dohmen

Infomatik-Bericht Nr. 114

INFORMATIK-BERICHTE

Herausgeber: Professoren des Institutes für Informatik
Redaktion: Publikationsstelle, Tel. (+49 30) 20181-239
Druckerei: Humboldt-Universität zu Berlin
ISSN: 0863 - 095

Die Reihe Informatik-Berichte erscheint aperiodisch.

Dr. Klaus Dohmen
Humboldt-Universität zu Berlin
Institut für Informatik
Sitz: Rudower Chausee 5
Unter den Linden 6
10099 Berlin
e-mail: dohmen@informatik.hu-berlin.de

Oktober 1998

Improved Bonferroni Bounds for the Reliability of a Communication Network

Klaus Dohmen
Humboldt-Universität zu Berlin
Institut für Informatik
Unter den Linden 6
D-10099 Berlin, Germany
e-mail: dohmen@informatik.hu-berlin.de

AMS Classification: 60C05, 68M15, 90B12, 90B25

Abstract

We present a new improvement upon the classical Bonferroni inequalities and show how this improvement can be utilized in bounding the reliability of a communication network whose nodes are perfectly reliable and whose edges are subject to random and independent failure.

1. Introduction

In this paper, we consider probabilistic networks whose nodes are perfectly reliable and whose edges are subject to random and independent failure where all failure probabilities are assumed to be known. There are a lot of performance measures associated with such a network, e.g., the probability that each pair of nodes can communicate along a path of operating edges (the so-called *all-terminal reliability*) or the probability that a distinguished pair of nodes can communicate along a path of operating edges (the so-called *two-terminal reliability*). In order to unify the various concepts, one usually employs the notion of a coherent binary system, which, by definition, is a couple $\Sigma = (E, \phi)$ where E is a finite set (the *component set* of Σ) and ϕ is a function (the *structure function* of Σ) from the power set of E into $\{0; 1\}$ such that $\phi(\emptyset) = 0$, $\phi(E) = 1$ and $\phi(X) \leq \phi(Y)$ for any $X, Y \subseteq E$ with $X \subseteq Y$ and where at any instant of time, each component $e \in E$ assumes randomly and independently one of two states, *operating* or *failing*, with probabilities p_e and $q_e = 1 - p_e$, respectively. We say that Σ is operating resp. failing if ϕ applied to the set of operating components equals 1 resp. 0. Since the probability that Σ is operating is determined by Σ and $\mathbf{p} = (p_e)_{e \in E}$, it is abbreviated to $\mathrm{Rel}_\Sigma(\mathbf{p})$. A key role in computing $\mathrm{Rel}_\Sigma(\mathbf{p})$ is played by the minpaths and mincuts of Σ: A *minpath* resp. *mincut* of Σ is a minimal subset F of E such that $\phi(F) = 1$ resp. $\phi(E \setminus F) = 0$.

For all-terminal reliability of an undirected network, for instance, the appropriate model is a coherent binary system $\Sigma = (E, \phi)$, where E is the edge-set of the network and $\phi(X) = 1$ if and only if the subnetwork induced by X is connected. Note that in this case, the minpaths correspond to the spanning trees and the mincuts to the cutsets (= minimal edge disconnecting sets) of the network.

Instead of exactly computing the reliability of a coherent binary system, which is known to be computationally intractible, one often applies Monte-Carlo or bounding techniques. Many of these techniques make use of the classical Bonferroni inequalities. For any probability space $(\Omega, \mathcal{A}, \mu)$, any finite collection of events $\{A_v\}_{v\in V} \subseteq \mathcal{A}$ and any $n \in \mathbb{N}$ the classical Bonferroni inequalities state that

$$\mu\left(\bigcup_{v\in V} A_v\right) \geq \sum_{\substack{I\subseteq V, I\neq\emptyset \\ |I|\leq n}} (-1)^{|I|-1} \mu\left(\bigcap_{i\in I} A_i\right) \qquad (n \text{ even}),$$

$$\mu\left(\bigcup_{v\in V} A_v\right) \leq \sum_{\substack{I\subseteq V, I\neq\emptyset \\ |I|\leq n}} (-1)^{|I|-1} \mu\left(\bigcap_{i\in I} A_i\right) \qquad (n \text{ odd}).$$

Recently, Naiman and Wynn [3] (see also [4]) introduced a framework for improving and generalizing the classical Bonferroni inequalities. In Section 2, we review this framework and establish a new improvement upon the classical Bonferroni inequalities. In Section 3, this improvement is used to obtain new bounds for the reliability of a coherent binary system, which generalize Shier's [5, 6] exact expression for the reliability of a coherent binary system as an alternating sum over chains in a semilattice. Finally, the new bounds are applied to a concrete communication network.

2. Improved Bonferroni inequalities

We start with some simple terminologies and facts from combinatorial topology. An *abstract simplicial complex* $\mathfrak{S}$ is a set of non-empty subsets of some finite set such that $I \in \mathfrak{S}$ and $\emptyset \neq J \subseteq I$ imply $J \in \mathfrak{S}$. The elements of $\mathfrak{S}$ resp. $\bigcup \mathfrak{S} = \bigcup_{I\in\mathfrak{S}} I$ are the *faces* resp. *vertices* of $\mathfrak{S}$. For any non-empty subset J of the vertices of $\mathfrak{S}$, we use $\mathfrak{S}(J)$ to denote the abstract simplicial complex containing all faces of $\mathfrak{S}$ which are included by J as a subset. By definition, the *dimension* of a face I, denoted by $\dim I$, is one less than its cardinality. The *dimension* of $\mathfrak{S}$, denoted by $\dim \mathfrak{S}$, is the maximum dimension of a face in $\mathfrak{S}$. The *Euler characteristic* of $\mathfrak{S}$ is defined by

$$\chi(\mathfrak{S}) := \sum_{I\in\mathfrak{S}} (-1)^{\dim I}.$$

For example, the abstract simplicial complex $\mathfrak{P}^*(V)$ consisting of all non-empty subsets of some finite non-empty set V has Euler characteristic 1.

Proposition 2.1 (Naiman-Wynn [3]). *Let $(\Omega, \mathcal{A}, \mu)$ be a probability space, $\{A_v\}_{v\in V} \subseteq \mathcal{A}$ be a finite collection of events and $\mathfrak{S} \subseteq \mathfrak{P}^*(V)$ be an abstract simplicial* *complex. If $\mathfrak{S}(J)$ has Euler characteristic ≤ 1 for all non-empty subsets J of the vertex-set of $\mathfrak{S}$ satisfying $\bigcap_{j\in J} A_j \cap \bigcap_{j\notin J} \complement A_j \neq \emptyset$, then*

$$\mu\left(\bigcup_{v\in V} A_v\right) \geq \sum_{I\in\mathfrak{S}} (-1)^{|I|-1}\,\mu\left(\bigcap_{i\in I} A_i\right).$$

On the other hand, if $\mathfrak{S}(J)$ has Euler characteristic ≥ 1 for all non-empty subsets J of the vertex-set of $\mathfrak{S}$ satisfying $\bigcap_{j\in J} A_j \cap \bigcap_{j\notin J} \complement A_j \neq \emptyset$, then

$$\mu\left(\bigcup_{v\in V} A_v\right) \leq \sum_{I\in\mathfrak{S}} (-1)^{|I|-1}\,\mu\left(\bigcap_{i\in I} A_i\right).$$

Lemma 2.2. *Let $\mathfrak{S}$ be an abstract simplicial complex, and let x be a vertex of $\mathfrak{S}$ such that $I \in \mathfrak{S}$ and $\dim I < \dim \mathfrak{S}$ imply $I \cup \{x\} \in \mathfrak{S}$. Then, $\chi(\mathfrak{S}) \leq 1$ if the dimension of $\mathfrak{S}$ is odd, and $\chi(\mathfrak{S}) \geq 1$ if the dimension of $\mathfrak{S}$ is even.*

Proof. By the definition of the Euler characteristic,

$$\begin{aligned}\chi(\mathfrak{S}) &= 1 + \sum_{I\in\mathfrak{S},\, x\notin I} (-1)^{\dim I} + \sum_{\substack{I\in\mathfrak{S},\, x\in I\\ I\neq\{x\}}} (-1)^{\dim I}\\ &= 1 + \sum_{\substack{I\in\mathfrak{S},\, x\notin I\\ \dim I=\dim\mathfrak{S}}} (-1)^{\dim I} + \sum_{\substack{I\in\mathfrak{S},\, x\notin I\\ \dim I<\dim\mathfrak{S}}} (-1)^{\dim I} + \sum_{\substack{I\in\mathfrak{S},\, x\notin I\\ \dim I<\dim\mathfrak{S}}} (-1)^{\dim(I\cup\{x\})}.\end{aligned}$$

Since $\dim(I\cup\{x\}) = \dim I + 1$ for any $x \notin I$, the last two sums cancel. Therefore,

$$\chi(\mathfrak{S}) = 1 + \sum_{\substack{I\in\mathfrak{S},\, x\notin I\\ \dim I=\dim\mathfrak{S}}} (-1)^{\dim I} = 1 + (-1)^{\dim\mathfrak{S}} \sum_{\substack{I\in\mathfrak{S},\, x\notin I\\ \dim I=\dim\mathfrak{S}}} 1\,. \blacksquare$$

Lemma 2.3. *Let $\mathfrak{X}$ be a set of non-empty subsets of some finite set V such that $V \neq \bigcup\mathfrak{X}$, and for any $n \in \mathbb{N}$ define*

$$\mathfrak{I}_n(V,\mathfrak{X}) := \{I \subseteq V : 0 < |I| \leq n,\, I \not\supseteq X\ \forall X \in \mathfrak{X}\}. \tag{1}$$

Then, $\chi(\mathfrak{I}_n(V,\mathfrak{X})) \leq 1$ if n is even, and $\chi(\mathfrak{I}_n(V,\mathfrak{X})) \geq 1$ if n is odd.

Proof. Choose $x \in V \setminus \bigcup\mathfrak{X}$ and apply Lemma 2.2. $\blacksquare$

The first main result of this paper is the following:

Theorem 2.4. *Let $(\Omega, \mathcal{A}, \mu)$ be a probability space and $\{A_v\}_{v\in V} \subseteq \mathcal{A}$ be a finite collection of events. In addition, let V be equipped with a partial ordering relation, and let $\mathfrak{X}$ be a set of non-empty subsets of V such that for any $X \in \mathfrak{X}$,*

$$\bigcap_{x\in X} A_x \subseteq \bigcup_{x\in X'} A_x\,, \tag{2}$$

where X' is the set of lower bounds of X which are not contained in X, that is,

$$X' := \{v \in V : v < x \; \forall x \in X\}.$$

Then, for any $n \in \mathbb{N}$,

$$\mu\left(\bigcup_{v\in V} A_v\right) \geq \sum_{I\in\mathfrak{I}_n(V,\mathfrak{X})} (-1)^{|I|-1} \mu\left(\bigcap_{i\in I} A_i\right) \quad (n \text{ even}),$$

$$\mu\left(\bigcup_{v\in V} A_v\right) \leq \sum_{I\in\mathfrak{I}_n(V,\mathfrak{X})} (-1)^{|I|-1} \mu\left(\bigcap_{i\in I} A_i\right) \quad (n \text{ odd}),$$

where $\mathfrak{I}_n(V,\mathfrak{X})$ is defined as in Eq. (1).

Proof. We restrict ourselves to the case where n is even. By Proposition 2.1 we only have to show that $\chi(\mathfrak{I}_n(V,\mathfrak{X})(J)) \leq 1$ for each non-empty subset J of the vertex-set of $\mathfrak{I}_n(V,\mathfrak{X})$ satisfying $\bigcap_{j\in J} A_j \cap \bigcap_{j\notin J} \complement A_j \neq \emptyset$. Suppose $\chi(\mathfrak{I}_n(V,\mathfrak{X})(J)) > 1$ for some J of the specified type. We first observe that $\mathfrak{I}_n(V,\mathfrak{X})(J) = \mathfrak{I}_n(J, \mathfrak{X}\cap\mathfrak{P}^*(J))$, and then, by Lemma 2.3, we conclude that $J = \bigcup(\mathfrak{X}\cap\mathfrak{P}^*(J))$. Hence, there is some $X \in \mathfrak{X}$ such that $X \subseteq J$ and $\min X = \min J$, and from this it follows that

$$\bigcap_{j\in J} A_j \subseteq \bigcap_{j\in X} A_j \subseteq \bigcup_{j\in X'} A_j \subseteq \bigcup_{j\notin J} A_j.$$

Therefore, $\bigcap_{j\in J} A_j \cap \bigcap_{j\notin J} \complement A_j = \emptyset$, which is contradiction with the choice of J. ▮

Remark. Note that for $n = |V|$ the inequalities in Theorem 2.4 become an identity. The identity corresponding to the special case where any $X \in \mathfrak{X}$ satisfies $\bigcap_{x\in X} A_x \subseteq A_{x'}$ for some $x' \in X'$ has been published in [1] and applied to network reliability problems in [2]. The inequalities, even those corresponding to this special case, are new. We further remark that in Theorem 2.4, $\mathfrak{X}$ does not need to consist of *all* non-empty subsets of V that satisfy Eq. (2). In the extreme case where $\mathfrak{X}$ is empty, the inequalities in Theorem 2.4 are just the classical Bonferroni inequalities.

3. Improved bounds for the reliability of a network

We now state the second main result of this paper. Recall that a chain in a partially ordered set is a subset containing no two incomparable elements.

Theorem 3.1. *Let $\Sigma = (E, \phi)$ be a coherent binary system, $\mathcal{S}$ the set of minpaths resp. mincuts of Σ and $\mathcal{T}$ an upper set of $\mathcal{S}$ such that each $T \in \mathcal{T}$ is an upper set of some $S \in \mathcal{S}$. Further, let $\mathcal{T}$ be equipped with a partial ordering relation such that for any $T_1, T_2 \in \mathcal{T}$, $T \subseteq T_1 \cup T_2$ for some lower bound T of T_1 and T_2. Then, in case* *that $\mathcal{S}$ denotes the set of minpaths,*

$$\mathrm{Rel}_\Sigma(\mathbf{p}) \geq \sum_{\substack{\mathcal{I}\in\mathrm{chains}(\mathcal{T})\\ 0<|\mathcal{I}|\leq n}} (-1)^{|\mathcal{I}|-1} \prod_{e\in\bigcup\mathcal{I}} p_e \quad (n \text{ even}), \tag{3}$$

$$\mathrm{Rel}_\Sigma(\mathbf{p}) \leq \sum_{\substack{\mathcal{I}\in\mathrm{chains}(\mathcal{T})\\ 0<|\mathcal{I}|\leq n}} (-1)^{|\mathcal{I}|-1} \prod_{e\in\bigcup\mathcal{I}} p_e \quad (n \text{ odd}), \tag{4}$$

and in case that $\mathcal{S}$ denotes the set of mincuts,

$$1-\mathrm{Rel}_\Sigma(\mathbf{p}) \geq \sum_{\substack{\mathcal{I}\in\mathrm{chains}(\mathcal{T})\\ 0<|\mathcal{I}|\leq n}} (-1)^{|\mathcal{I}|-1} \prod_{e\in\bigcup\mathcal{I}} q_e \quad (n \text{ even}), \tag{5}$$

$$1-\mathrm{Rel}_\Sigma(\mathbf{p}) \leq \sum_{\substack{\mathcal{I}\in\mathrm{chains}(\mathcal{T})\\ 0<|\mathcal{I}|\leq n}} (-1)^{|\mathcal{I}|-1} \prod_{e\in\bigcup\mathcal{I}} q_e \quad (n \text{ odd}), \tag{6}$$

where throughout chains($\mathcal{T}$) *denotes the set of chains in $\mathcal{T}$ and $n \in \mathbb{N}$.*

Proof. Let μ denote the induced probability measure on the set of network states, and for any $T \in \mathcal{T}$, let A_T denote the event that all components in T are operating (in case that $\mathcal{S}$ denotes the set of minpaths) resp. failing (in case that $\mathcal{S}$ denotes the set of mincuts). Then,

$$\mathrm{Rel}_\Sigma(\mathbf{p}) = \mu\left(\bigcup_{T\in\mathcal{T}} A_T\right) \quad \text{resp.} \quad 1-\mathrm{Rel}_\Sigma(\mathbf{p}) = \mu\left(\bigcup_{T\in\mathcal{T}} A_T\right).$$

Moreover, it is easy to verify that

$$\mu\left(\bigcap_{I\in\mathcal{I}} A_I\right) = \prod_{e\in\bigcup\mathcal{I}} p_e \quad \text{resp.} \quad \mu\left(\bigcap_{I\in\mathcal{I}} A_I\right) = \prod_{e\in\bigcup\mathcal{I}} q_e.$$

In both cases, the assumptions imply that for each two incomparable $T_1, T_2 \in \mathcal{T}$ there is some $T \in \mathcal{T}$ strictly less than T_1 and T_2 such that $A_{T_1} \cap A_{T_2} \subseteq A_T$. Therefore, the result immediately follows by setting $\mathfrak{X}$ equal to the set of all incomparable elements of $\mathcal{T}$ and then applying Theorem 2.4. ▮

Remarks. For $n = |\mathcal{T}|$ the above inequalities give rise to the following identities:

$$\mathrm{Rel}_\Sigma(\mathbf{p}) = \sum_{\substack{\mathcal{I}\in\mathrm{chains}(\mathcal{T})\\ \mathcal{I}\neq\emptyset}} (-1)^{|\mathcal{I}|-1} \prod_{e\in\bigcup\mathcal{I}} p_e \quad \text{resp.} \quad 1-\mathrm{Rel}_\Sigma(\mathbf{p}) = \sum_{\substack{\mathcal{I}\in\mathrm{chains}(\mathcal{T})\\ \mathcal{I}\neq\emptyset}} (-1)^{|\mathcal{I}|-1} \prod_{e\in\bigcup\mathcal{I}} q_e.$$

In a completely different way, these identities were first established by Shier [5, 6]. The bounds of Theorem 3.1, however, are new. In many important situations (e.g., if $\mathcal{T}$ equals $\mathcal{S}$ and is not a chain) these new bounds are much sharper than the classical bounds and require much fewer sets to be inspected during their computation.

We finally consider the particular case of bounding the all-terminal reliability of a directed or undirected network whose nodes are perfectly reliable and whose edges fail randomly and independently with known probabilities. Let N, E and $\mathcal{S}$ denote the set of nodes, edges and cutsets of the network, respectively, and for any non-empty proper subset M of N let $\langle M \rangle$ consist of all edges linking some node in M to some node outside M. Then

$$\mathcal{T} := \{\langle M \rangle : \emptyset \neq M \subset N\}$$

is an upper set of $\mathcal{S}$, and moreover, each $T \in \mathcal{T}$ is an upper set of some $S \in \mathcal{S}$. (Note that $\mathcal{T} = \mathcal{S}$ if and only if the network is complete.) Now, fix some $a \in N$, and for any non-empty proper subsets M_1, M_2 of N containing a define

$$\langle M_1 \rangle \preceq \langle M_2 \rangle \quad :\Longleftrightarrow \quad M_1 \subseteq M_2 \,.$$

Then, $\langle M_1 \cap M_2 \rangle$ is the greatest lower bound of $\langle M_1 \rangle$ and $\langle M_2 \rangle$, and

$$\langle M_1 \cap M_2 \rangle \subseteq \langle M_1 \rangle \cup \langle M_2 \rangle \,.$$

Hence, the requirements of Theorem 3.1 are satisfied, and thus Eqs. (5) and (6) give upper resp. lower bounds for the all-terminal reliability of the network. (Partial ordering relations appropriate for computing or bounding the two-terminal reliability of a directed or undirected network can be adapted from Shier [5, 6]; see also [2].)

Example. Consider the network in Figure 1. For this network, the Hasse diagram of $(\mathcal{T}, \preceq)$ is shown in Figure 2. Under the assumption that all edges fail with the same probability $q = 1-p$, Table 1 shows the classical bounds $b_n(q)$ $(n = 1, 2, \dots)$ and the improved bounds $b_n^*(q)$ $(n = 1, 2, \dots)$ for the all-terminal reliability of the network together with the number of sets inspected during the computation of each bound. A numerical comparism of classical and improved bounds in shown in Table 2.

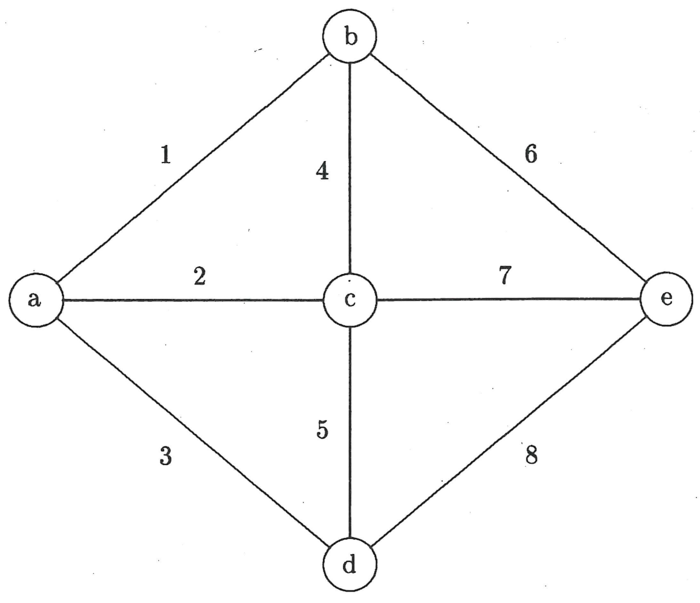

Figure 1: A sample network.

n	classical bounds $b_n(q)$	# sets	improved bounds $b_n^*(q)$	# sets
1	$1-4q^3-5q^4-4q^5$	13	$1-4q^3-5q^4-4q^5-2q^6$	15
2	$1-4q^3-5q^4+8q^5+44q^6+20q^7+2q^8$	91	$1-4q^3-5q^4+4q^5+30q^6+8q^7+2q^8$	65
3	$1-4q^3-5q^4+4q^5+30q^6-160q^7-86q^8$	377	$1-4q^3-5q^4+4q^5+30q^6-40q^7-10q^8$	125
4	$1-4q^3-5q^4+4q^5+30q^6+40q^7+429q^8$	1092	$1-4q^3-5q^4+4q^5+30q^6-40q^7+14q^8$	149
5	$1-4q^3-5q^4+4q^5+30q^6-68q^7-750q^8$	2379		
6	$1-4q^3-5q^4+4q^5+30q^6-36q^7+934q^8$	4095		
7	$1-4q^3-5q^4+4q^5+30q^6-40q^7-778q^8$	5811		
8	$1-4q^3-5q^4+4q^5+30q^6-40q^7+509q^8$	7098		
9	$1-4q^3-5q^4+4q^5+30q^6-40q^7-206q^8$	7813		
10	$1-4q^3-5q^4+4q^5+30q^6-40q^7+80q^8$	8099		
11	$1-4q^3-5q^4+4q^5+30q^6-40q^7+2q^8$	8177		
12	$1-4q^3-5q^4+4q^5+30q^6-40q^7+15q^8$	8190		
13	$1-4q^3-5q^4+4q^5+30q^6-40q^7+14q^8$	8191		

Table 1: Bounds for the all-terminal reliability of the network in Figure 1.

q	$b_1(q)$	$b_1^*(q)$	$b_3(q)$	$b_3^*(q)$	$b_5(q)$	$b_5^*(q)^\dagger$	$b_4^*(q)^\dagger$	$b_4(q)$	$b_2^*(q)$	$b_2(q)$
0.00	1.00000	1.00000	1.00000	1.00000	1.00000	*1.00000*	*1.00000*	1.00000	1.00000	1.00000
0.05	0.99947	0.99947	0.99947	0.99947	0.99947	*0.99947*	*0.99947*	0.99947	0.99947	0.99947
0.10	0.99546	0.99546	0.99555	0.99557	0.99556	*0.99557*	*0.99557*	0.99558	0.99557	0.99563
0.15	0.98367	0.98364	0.98432	0.98454	0.98431	*0.98455*	*0.98455*	0.98479	0.98463	0.98511
0.20	0.95872	0.95859	0.96093	0.96266	0.96041	*0.96272*	*0.96272*	0.96481	0.96331	0.96564
0.25	0.91406	0.91357	0.91812	0.92661	0.91361	*0.92697*	*0.92697*	0.93819	0.92972	0.93778
0.30	0.84178	0.84032	0.84246	0.87369	0.81901	*0.87526*	*0.87526*	0.91999	0.88497	0.90752
0.35	0.73246	0.72878	0.70732	0.80164	0.61698	*0.80704*	*0.80704*	0.95197	0.83522	0.88969
0.40	0.57504	0.56685	0.46134	0.70775	0.17691	*0.72348*	*0.72348*	1.12653	0.79426	0.91222
0.45	0.35666	0.34005	0.01091	0.58711	-0.76184	*0.62747*	*0.62747*	1.62423	0.78665	1.02155
0.50	0.06250	0.03125	-0.80469	0.42969	-2.67969	*0.52344*	*0.52344*	2.76953	0.85156	1.28906

†exact network reliability

Table 2: Numerical values of the bounds in Table 1.

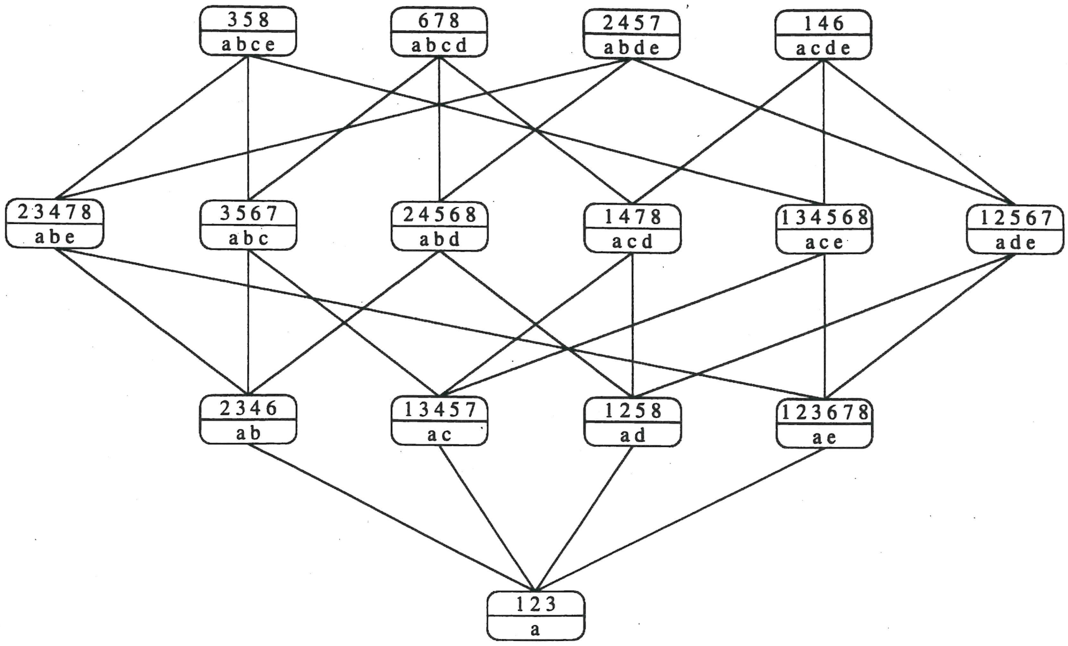

Figure 2: Hasse diagram for the network in Figure 1.

References

[1] K. DOHMEN, A note on Möbius inversion over power set lattices, *Commentat. Math. Univ. Carol.* **38** (1997), 121–124.

[2] K. DOHMEN, Inclusion-exclusion and network reliability, *Electron. J. Combin.* **5** (1998), paper no. 36. [Internet: `http://www.combinatorics.org`]

[3] D.Q. NAIMAN and H.P. WYNN, Inclusion-exclusion-Bonferroni identities and inequalities for discrete tube-like problems via Euler characteristics, *Ann. Statist.* **20** (1992), 43–76.

[4] D.Q. NAIMAN and H.P. WYNN, Abstract tubes, improved inclusion-exclusion identities and inequalities and importance sampling, *Ann. Statist.* **25** (1997), 1954–1983.

[5] D.R. SHIER, Algebraic aspects of computing network reliability, *Applications of Discrete Mathematics* (ed. R.D. RINGEISEN and F.S. ROBERTS), SIAM, Philadelphia, 1988, 135–147.

[6] D.R. SHIER, *Network reliability and algebraic structures*, Clarendon Press, Oxford, 1991.